Renormalization of the charged scalar field in curved space

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Abstract

The DeWitt-Schwinger proper time point-splitting procedure is applied to a massive complex scalar field with arbitrary curvature coupling interacting with a classical electromagnetic field in a general curved spacetime. The scalar field current is found to have a linear divergence. The presence of the external background gauge field is found to modify the stress-energy tensor results of Christensen for the neutral scalar field by adding terms of the form $(eF)^2$ to the logarithmic counterterms. These results are shown to be expected from an analysis of the degree of divergence of scalar quantum electrodynamics.

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I. INTRODUCTION

In the study of quantized fields in curved spacetimes, the use of geodesic point-splitting as a regularization method for quantities such as the vacuum polarization $\langle \phi^2 \rangle$ and the stress-energy tensor $\langle T_{\mu\nu} \rangle$ has been shown to be a robust and trustworthy method [1]. The detailed method of point-splitting regularization was developed from the DeWitt-Schwinger proper time method of calculating the Feynman Green function by Christensen [2,3] for the stress-energy tensor of a massive real scalar field. In this paper, we extend Christensen’s work to develop the point-splitting counterterms necessary to regularize the current four-vector and stress-energy tensor for a massive complex charged scalar field with arbitrary curvature coupling in a general curved spacetime background with a nonzero background classical electromagnetic field. These results will allow, for the first time, calculation of both the expectation value of the current, $\langle j^\mu \rangle$, and the stress-energy tensor, $\langle T_{\mu\nu} \rangle$, of a charged quantized field in a fixed background spacetime containing an electromagnetic field. In addition, our results may also be used to consider the combined Einstein-Maxwell semiclassical backreaction problem, where the gravitational and electromagnetic fields are treated classically, and their sources are taken to be quantized fields:

$$G_{\mu\nu} = 8\pi \langle T_{\mu\nu} \rangle, \quad (1)$$

and,

$$F^{\mu\nu} = 4\pi \langle j^\mu \rangle. \quad (2)$$

Our primary motivation for this work is to apply these results in the study of charged quantized fields in the spacetime of a charged spherical black hole, which would be described classically by the Reissner-Nordström metric. The electromagnetic field of the Reissner-Nordström black hole gives rise to such exotic structures in the black hole interior as Cauchy horizons, timelike singularities, and an apparent tunnel to other asymptotically flat space-times.
The key physical question is whether the analytically extended Reissner-Nordström solution correctly describes the interior of a charged spherical black hole. The Cauchy horizon has been shown to be both classically \cite{9,10} and quantum mechanically unstable \cite{11,12}. The metric backreaction to the classical instability has been shown to result in the Cauchy horizon becoming a curvature singularity, via the so-called ”mass inflation” process \cite{13,14}. It is, however, somewhat unclear whether this singularity is sufficient enough to act as a true “edge” to the spacetime, enforcing strong cosmic censorship \cite{15,16}.

The studies of Cauchy horizon instability and mass inflation deal with the evolution of uncharged fields on an initially Reissner-Nordström background. Yet, since it is the existence of the electromagnetic field which causes the exotic internal structures such as the Cauchy horizons to exist, one should surely study the stability and evolution of the electromagnetic field in these solutions. In particular, since extremely strong electric or magnetic fields will be encountered in the black hole interior, it is appropriate to study the pair creation and vacuum polarization of charged quantized fields in these spacetimes. Gibbons \cite{17} has studied the effects of thermal particle production on charged black holes, while Davies \cite{18} has considered the thermodynamic implications of how the charge of the black hole affects its evaporation. However, both of these studies were performed in the context of a fixed background (constant mass, $M$, and charge, $Q$) spacetime.

Some studies have been performed wherein the electromagnetic field of the black hole is allowed to create charged pairs whose field serves to modify the electromagnetic field that created them. Hiscock and Weems \cite{19} considered the loss of charge and mass for the exteriors of charged black holes of large initial mass. They assumed adiabatic evolution, so that the spacetime could be described by a sequence of Reissner-Nordström metrics with the mass and charge of the black hole being slowly varying functions of time. Integrating these equations, they investigated the dynamical future of the discharging, evaporating black hole, but their treatment was limited to the region exterior to the event horizon.

Several studies have attempted to model the evolution of the interior electromagnetic field and metric. Novikov and Starobinskiĭ \cite{20} have studied the dynamical evolution of the
electric field within the outer horizon. In their model, the electric field serves as the source of its own demise through the production of electron-positron pairs. The effect of these pairs on the interior spacetime is such that it seems to evolve from an initial Reissner-Nordström state to a final state which is uncharged and (in broad terms) Schwarzschild-like. Their model assumes an initial Reissner-Nordström geometry on a spacelike slice inside the outer event horizon with the pair production being allowed to modify the electric field to the future of that initial surface.

So far these studies have been done using many simplifying assumptions. These assumptions, while giving intuitively satisfying results, are still approximations to the true dynamical equations. They have often used the Schwinger formula \[ \text{[21]} \] for the number of charged pairs created per unit four-volume of spacetime as the source of their pair creation. This formula was derived with its own simplifying assumptions of a static and uniform electric field, and a flat background spacetime. And while it is considered valid in the regime of small curvature and slowly varying electromagnetic fields, it is still, in essence, a flat-space result.

Clearly, a more complete treatment of the problem would involve choosing an initial spacelike surface consistent with an asymptotic Reissner-Nordström spacetime. Dynamical equations would then be allowed to evolve these initial conditions into the future, determining the true geometry through the use of both the semi-classical Einstein field equations and the Maxwell equations, Eqs.(1-2). In this context the problem of the Cauchy horizon would change from a question of whether it is unstable to some perturbation into a question of whether it is formed at all in a self-consistent treatment including semiclassical effects.

Recently, Anderson, Hiscock, and Samuel \[ \text{[22]} \] have described a method for numerically computing the vacuum expectation value of the stress tensor for quantized scalar fields in static spherically symmetric spacetimes with arbitrary mass and curvature coupling. Their method is fully renormalized in that they combine the DeWitt-Schwinger counterterms for the stress tensor of the real scalar field derived by Christensen \[ \text{[2]} \] with the expressions for the components of the unrenormalized stress tensor. With this renormalization, their
computational scheme may be carried to arbitrary numerical precision. This method is completely general and may be extended to the calculation of the expectation value of the current due to creation of charged pairs by the electromagnetic field of the charged black hole.

To begin the study of the semiclassical dynamics of a charged spherical black hole interior, in this paper we develop the theory of point-splitting regularization of the four-current vector and stress-energy tensor of a charged massive complex scalar field with arbitrary curvature coupling. The background spacetime is arbitrary, as is the background classical electromagnetic field. The complex scalar field is chosen for study in order to better make contact with existing point-splitting results, which have largely been derived only for scalar fields, and to avoid (for the present) the added complications of having to deal with spin.

In Sec. II, the equations for the expectation values of the current and stress-energy tensor equations are presented in the form required for the point-splitting procedure, revealing the Hadamard elementary function, $G^{(1)}(x, x')$, and its gauge covariant derivatives as the key quantities to analyze. In Sec. III, the ansatz of DeWitt used by Christensen is used to yield recursion relations of the familiar form. The presence of the gauge field in this case is seen to modify the original recursion relations such that they assume a gauge invariant form. Sec. IV contains a listing of the necessary coincidence limits required in the point-splitting expansions of the bitensors involved in the vacuum expectation value (VEV) of the current and stress tensor. Sec. V then uses these coincidence limits to expand the various bitensors appearing in $G^{(1)}(x, x')$ and its derivatives in power series about the fixed point $x$. The series involve powers of the separation vector connecting the point $x$ to the nearby point $x'$. In Sec. VI, the quantities derived in the preceding sections are combined to construct the divergences of the VEV of the current and the stress tensor. The complex scalar field current is seen to have a linear divergence, while the presence of the electromagnetic field adds solely to the logarithmic divergence of the stress tensor. Sec. VII briefly discusses why these results are consistent with those expected from an analysis of scalar quantum electrodynamics.
II. SEMI-CLASSICAL POINT-SEPARATED CURRENT AND STRESS TENSOR

A complex scalar field coupled to the electromagnetic field in an arbitrary curved background is described by the action functional \[ \mathcal{S}(\phi, A_\mu) = -\frac{1}{2} \int g^{\frac{1}{2}} \left[ (D_\mu \phi)(D^\mu \phi)^* + (m^2 + \xi R)\phi \phi^* - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right] d^4x, \] (3)

where \( \phi(x) = \frac{1}{\sqrt{2}}(\phi_1(x) + i\phi_2(x)) \) is the complex scalar field, \( g \) is the negative of the determinant of the metric \( g_{\mu\nu} \), \( D_\mu \equiv (\nabla_\mu - ieA_\mu) \) is the gauge covariant derivative, \( A^\mu \) is the classical electromagnetic vector potential, \( e \) is the coupling between the complex scalar and the electromagnetic fields, \( m \) is the mass of the complex scalar field, \( \xi \) is the scalar curvature coupling, and \( R \) is the scalar curvature. The wave equation obtained by varying the action in Eq. (3) with respect to \( \phi^* \) is

\[ \phi_{\mu\nu}^* - (m^2 + \xi R)\phi^* = 0, \] (4)

where the vertical slash denotes gauge covariant differentiation. The classical current equation obtained from Eq. (3) is

\[ j^\mu \equiv \frac{1}{4\pi} F^{\mu\nu} = \frac{1}{2}ie \left[ \{D^\mu \phi, \phi^*\} - \{D^\mu \phi, \phi^*\}^* \right], \] (5)

where the braces \( \{ \} \) denote the anticommutator. The components of the classical stress-energy tensor are given by,

\[ T^{\mu\nu} = \frac{1}{2} \{ \frac{1}{2} (1 - 2\xi) \{ \phi^{\mu\nu}, \phi^{\ast\nu} \} + \frac{1}{2} (2\xi - \frac{1}{2}) g^{\mu\nu} \{ \phi^{\nu\varphi}, \phi^{\ast\varphi} \} \\
- \xi \{ \phi^{\mu\nu}, \phi^{\ast} \} + \xi g^{\mu\nu} \{ \phi^{\nu\varphi}, \phi^{\ast} \} \\
+ \frac{1}{2} \xi (R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R) \{ \phi, \phi^{\ast} \} - \frac{1}{4} m^2 g^{\mu\nu} \{ \phi, \phi^{\ast} \} + c.c. \}, \] (6)

where \( c.c. \) denotes the complex conjugate of all of the previous terms. The anti-commutators above arise from symmetrizing with respect to the fields \( \phi \) and \( \phi^* \). As usual, the transition from classical to quantum fields is made by replacing each classical field \( \phi(x) \) with a field operator \( \hat{\phi}(x) \). Then, following Christensen, the first field operator \( \hat{\phi}(x) \) in each bracket in
Eq. (5) is moved to the nearby point $x'$ and evaluated between the vacuum states $\langle \text{out}, \text{vac} \rvert$ and $\rvert \text{in}, \text{vac} \rangle$. A similar expression is then obtained by taking the quantum version of Eq. (5) and moving the second field operator to the point $x'$. The two results are then averaged.

The Hadamard function $G^{(1)}$ and its first few gauge covariant derivatives may be written as

$$G^{(1)}(x, x') \equiv \frac{\langle \text{out}, \text{vac} \rvert \{\phi(x), \phi^*(x')\} \rvert \text{in}, \text{vac} \rangle}{\langle \text{out}, \text{vac} \rvert \text{in}, \text{vac} \rangle},$$  

(7a)

$$G^{(1)}_{\mu}(x, x') \equiv \frac{\langle \text{out}, \text{vac} \rvert \{\phi^{\mu}(x), \phi^*(x')\} \rvert \text{in}, \text{vac} \rangle}{\langle \text{out}, \text{vac} \rvert \text{in}, \text{vac} \rangle},$$  

(7b)

$$G^{(1)}_{\mu'}(x, x') \equiv \frac{\langle \text{out}, \text{vac} \rvert \{\phi(x), \phi^{*\mu'}(x')\} \rvert \text{in}, \text{vac} \rangle}{\langle \text{out}, \text{vac} \rvert \text{in}, \text{vac} \rangle},$$  

(7c)

$$G^{(1)}_{\mu \nu}(x, x') \equiv \frac{\langle \text{out}, \text{vac} \rvert \{\phi^{\mu \nu}(x), \phi^*(x')\} \rvert \text{in}, \text{vac} \rangle}{\langle \text{out}, \text{vac} \rvert \text{in}, \text{vac} \rangle},$$  

(7d)

$$G^{(1)}_{\mu \nu'}(x, x') \equiv \frac{\langle \text{out}, \text{vac} \rvert \{\phi(x), \phi^{*\mu \nu'}(x')\} \rvert \text{in}, \text{vac} \rangle}{\langle \text{out}, \text{vac} \rvert \text{in}, \text{vac} \rangle},$$  

(7e)

and

$$G^{(1)}_{\mu' \nu'}(x, x') \equiv \frac{\langle \text{out}, \text{vac} \rvert \{\phi(x), \phi^{*\mu' \nu'}(x')\} \rvert \text{in}, \text{vac} \rangle}{\langle \text{out}, \text{vac} \rvert \text{in}, \text{vac} \rangle}.  

(7f)$$

With these definitions, the expectation value of the scalar field current may be written in terms of the Hadamard function as

$$\langle j^\mu(x) \rangle = \lim_{x' \to x} \frac{ie}{4} \left[ (G^{(1)}_{\mu} + g^{\mu \tau} G^{(1)}_{\tau \nu'}) - (G^{(1)}_{\mu} + g^{\mu \tau} G^{(1)}_{\tau \nu'})^* \right].$$  

(8)

Note the current is guaranteed to be real. The expectation value of the stress-energy tensor may similarly be written as

$$\langle T^{\mu \nu}(x) \rangle = \lim_{x' \to x} \text{Re} \left[ \frac{1}{2} - \xi (g^{\mu \tau} G^{(1)}_{\tau \nu'}) + (\xi - \frac{1}{4}) g^{\mu \nu} g^{\alpha \rho} G^{(1)}_{\alpha \rho} \right] + \frac{1}{2} \xi (\nabla^{\mu} G^{(1)}_{\nu \rho} + g^{\nu \rho} G^{(1)}_{\mu \rho} - g^{\rho \mu} \nabla^{\nu}) $$  

$$- \frac{1}{2} \xi (G^{(1)}_{\mu \rho} + g^{\mu \rho} G^{(1)}_{\nu \rho'}) + \frac{1}{4} \xi g^{\mu \nu} (G^{(1)}_{\alpha \rho} + g^{\alpha \rho} G^{(1)}_{\nu \rho'}) $$  

$$+ \frac{1}{2} \xi (R^{\mu \nu} - \frac{1}{2} g^{\mu \nu} R) G^{(1)} - \frac{1}{4} m^2 g^{\mu \nu} G^{(1)} \right].$$  

(9)
Eqs. (8) and (9) are divergent; the following section will outline the point-splitting procedure for isolating the infinities of $G^{(1)}$ and its derivatives needed to regularize these expressions.

III. GREEN FUNCTIONS AND THE RECURSION RELATIONS

To isolate the infinities of the Hadamard elementary function $G^{(1)}(x, x')$ as the points are brought together, we use the relation,

$$ G_F(x, x') = -\frac{1}{2} i G^{(1)}(x, x'), $$

where $G_F(x, x')$ is the Feynman Green function, and $\overline{G}(x, x')$ is one-half the sum of the advanced and retarded Green functions. In coordinate space, the Feynman Green function satisfies,

$$ F(x)G_F(x, x') = -\delta(x - x'), $$

with the operator $F(x)$ given by $F(x) = g^{\frac{1}{4}} [D_\mu D^\mu - (m^2 + \xi R)]$; derivatives are with respect to $x$. After rewriting this as the matrix equation,

$$ \int \langle x | F | x'' \rangle \langle x'' | G | x' \rangle d^4 x'' = \langle x | -\mathbb{1} | x \rangle, $$

$F$ and $G$ are now matrix operators. Inserting two factors of $\mathbb{1} = g^{-\frac{1}{4}} g^\frac{1}{4}$ maintains the transformation properties of the matrix operators and allows the matrix equation to be written as,

$$ g^{\frac{1}{4}} G g^{-\frac{1}{4}} = \frac{1}{-\langle g^{\frac{1}{4}} F g^{-\frac{1}{4}} + i e \mathbb{1} \rangle} = i \int_0^\infty e^{-i H(s-0)} ds, $$

with the matrix operator $H \equiv -\langle g^{-\frac{1}{4}} F g^{\frac{1}{4}} + i e \mathbb{1} \rangle$. Taking matrix elements of Eq. (13) and rearranging, yields,

$$ G(x, x') = i \int_0^\infty g^{-\frac{1}{4}}(x) \langle x | e^{-i H(s-0)} | x' \rangle g^{-\frac{1}{4}}(x') ds $$

$$ \equiv i \int_0^\infty g^{-\frac{1}{4}}(x, s) | s | 0 \rangle g^{-\frac{1}{4}}(x') ds. $$

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The matrix element \( \langle x, s|x', 0 \rangle \) obeys the Schrödinger-like equation,

\[
\frac{i}{\partial s}\langle x, s|x', 0 \rangle = \langle x, s|H|x', 0 \rangle = g \frac{\pi}{\hbar} g^{-\frac{\pi}{\hbar}}(x') \left[ (m^2 + \xi R) - D_\mu D^\mu \right] \langle x, s|x', 0 \rangle, \tag{15}
\]

with the boundary condition,

\[
\langle x, 0|x', 0 \rangle = \langle x|x' \rangle = \delta(x - x'), \tag{16}
\]

where the infinitesimal factor \(+i\varepsilon\) has been dropped for brevity. We use the same ansatz as DeWitt [26] for the solution of Eq.(15),

\[
\langle x, s|x', 0 \rangle = \frac{i}{(4\pi)^2} \frac{D \frac{\pi}{\hbar} (x, x')}{s^2} \exp \left[ i\sigma(x, x') - \frac{im^2 s}{2s} \right] \Omega(x, x'), \tag{17}
\]

where the Van Vleck-Morette determinant is defined by \( D(x, x') \equiv -det(-\sigma_{\mu\nu}) \), and

\[
\sigma(x, x') \equiv \frac{1}{2} \sigma^{\mu\nu} \sigma_\mu = \frac{1}{2} \sigma^{\mu\nu} \sigma_\mu \tag{18}
\]

is one-half of the square of the geodesic distance between \( x \) and \( x' \). The vector \( \sigma^{\mu} \) is tangent to the geodesic at the point \( x \), has length equal to the geodesic distance between the points \( x \) and \( x' \), and points in the direction \( x' \to x \). We also use the identity [26],

\[
D^{-1} (D\sigma^{\mu})_{\|\mu} = 4. \tag{19}
\]

It should be noted that the scalars \( \sigma \) and \( D \) are geometric quantities unaffected by the presence of the gauge field. Thus their covariant derivatives will require the Christoffel connection but will not need any connection to the gauge field through \( A^\mu \).

Substituting Eq.(17) into Eq.(15), and using Eq.(19), a differential equation for \( \Omega(x, x') \) is obtained,

\[
\frac{i}{\partial s}\Omega + \frac{i}{s} \Omega_{\mu} \sigma^{\mu} = -D^{-\frac{1}{2}} (D^{\frac{1}{2}} \Omega)_{\|\mu} + \xi R \Omega. \tag{20}
\]

We now assume that \( \Omega \) may be represented by the power series,

\[
\Omega(x, x') = \sum_0^\infty a_n(x, x')(is)^n. \tag{21}
\]
This may be done, so long as the gravitational and electromagnetic fields, upon which
the coefficients \( a_n(x, x') \) depend, are slowly varying over the infinitesimal distance be-
tween the points \( x \) and \( x' \). Substituting Eq.\((22)\) into Eq.\((20)\), and defining
\( \Delta(x, x') \equiv g^{-\frac{1}{2}}(x)D(x, x')g^{-\frac{1}{2}}(x') \), yields the recursion relations which will be used to determine the \( a_n \),

\[
\sigma^{\mu} a_{0|\mu} = 0, \tag{22}
\]

and

\[
\sigma^{\mu} a_{n+1|\mu} + (n + 1) a_{n+1} = \Delta^{-\frac{1}{2}}(\Delta^{\frac{1}{2}} a_n)_{\mu} - \xi R a_n. \tag{23}
\]

These relations are of the same form as those derived by Christensen. The presence of the
gauge field in this case now requires that all derivatives be gauge covariant. As mentioned
previously, objects such as \( \sigma^{\mu}, \Delta^{\frac{1}{2}} R, R^{\mu\nu\rho}, \) etc., will only require the Christoffel connections.
Only the coefficients \( a_n(x, x') \), which carry information about the gauge field, will require
the gauge connection \( A^{\mu} \) in their derivatives.

Finally, after substituting Eq.\((21)\) and Eq.\((17)\) into Eq.\((14)\), and performing a few
straightforward but long mathematical manipulations which will not be detailed here (see
Refs. \([2,26]\) for details), we arrive at the final form for \( G^{(1)}(x, x') \), namely

\[
G^{(1)}(x, x') = \frac{\Delta^{\frac{1}{2}}}{4\pi^2} \left\{ a_0 \left[ \frac{1}{\sigma} + m^2 (\gamma + \frac{1}{2} \ln \left| \frac{1}{2} m^2 \sigma \right|) (1 + \frac{1}{4} m^2 \sigma + \cdots) - \frac{1}{2} m^2 - \frac{5}{16} m^2 \sigma + \cdots \right] 
- a_1 \left[ (\gamma + \frac{1}{2} \ln \left| \frac{1}{2} m^2 \sigma \right|) (1 + \frac{1}{2} m^2 \sigma + \cdots) - \frac{1}{2} m^2 \sigma - \cdots \right] 
+ a_2 \sigma \left[ (\gamma + \frac{1}{2} \ln \left| \frac{1}{2} m^2 \sigma \right|) \left( \frac{1}{2} + \frac{1}{8} m^2 \sigma + \cdots \right) - \frac{1}{4} - \cdots \right] 
+ \frac{1}{2m^2} a_2 + \cdots + \frac{1}{2m^4} [a_3 + \cdots] + \cdots \right\}. \tag{24}
\]

With the relations of Eq.\((22)\) and Eq.\((23)\) for the \( a_n \), the changes in \( G^{(1)}(x, x') \) due to the
presence of the gauge field will be carried through the \( a_n \). It is easily seen that \( G^{(1)} \) as
expressed in Eq.\((24)\) has at least a quadratic divergence in the infinitesimal separation \( \sigma^{\mu} \)
due to the presence of the term proportional to \( 1/\sigma = 2/(\sigma_{\mu\nu} \sigma^{\mu}) \). Isolation of this and other
divergences in \( G^{(1)}(x, x') \) and its derivatives are discussed in the next two sections.
IV. COINCIDENCE LIMITS

In this section the coincidence limits $x' \to x$ of Eqs. (18), (19), (22), and (23) and their derivatives are developed. Synge’s bracket notation [27],

$$\lim_{x' \to x} a(x, x') \equiv [a(x, x')]$$

will be used to simplify the writing of the many limits needed. The coincidence limits of $\sigma(x, x')$ and $\Delta^\downarrow(x, x')$ and their covariant derivatives are unaffected by the presence of the gauge field and hence are unchanged from the results of Christensen. He finds [3],

$$[\sigma^{\mu}] = 0,$$  
$$[\sigma^{\mu}_\nu] = g^{\mu}_\nu,$$  
$$[\sigma^{\mu}_\nu] = 0,$$  
$$[\sigma^{\mu}_\nu\rho\tau\rho\tau] = -\frac{1}{3}(R^{\mu}_{\nu\rho\tau} + R^{\mu}_{\tau\nu\rho}),$$  
$$[\sigma^{\mu}_\nu\rho\tau\rho\tau] = \frac{3}{4}(S^{\mu}_{\nu\rho\tau\rho\tau} + S^{\mu}_{\nu\rho\tau\rho\alpha} + S^{\mu}_{\nu\rho\alpha\tau\tau}),$$

and a six-derivative limit with 36 terms involving $S^{\mu}_{\nu\rho\tau}$ which may be found in Ref. [3]. Note that a semicolon has been used here to emphasize that the covariant derivatives contain only the Christoffel connections. Also,

$$[\Delta^\downarrow] = 1,$$  
$$[\Delta^\downarrow_{\mu\nu}] = 0,$$  
$$[\Delta^\downarrow_{\mu\nu\rho\tau\rho\tau}] = \frac{1}{6}R_{\mu\nu},$$  
$$[\Delta^\downarrow_{\mu\nu\rho\tau\rho\tau}] = \frac{1}{12}(R_{\mu\nu;\rho} + R_{\nu\rho;\mu} + R_{\rho\mu;\nu}),$$

and a four-derivative limit with 12 terms which will not be shown here.

Differentiating Eq. (22) repeatedly we find,

$$\sigma^{\mu}a_{0|\mu} = 0,$$  
$$\sigma^{\mu\nu}a_{0|\mu} + \sigma^{\mu}a_{0|\mu}^\nu = 0,$$  
$$\sigma^{\mu\nu\rho}a_{0|\mu} + \sigma^{\mu\nu}a_{0|\mu}^\rho + \sigma^{\mu\rho}a_{0|\mu}^\nu + \sigma^{\mu}a_{0|\mu}^\nu\rho = 0,$$
and so forth, where the slash is used to emphasize the gauge covariant derivative being applied to $a_0$. Taking the coincidence limit of Eqs.(35-37), and using Eqs. (26-30) along with the commutation relation of the gauge covariant derivative,

$$a_{n|\mu\nu} - a_{n|\nu\mu} = i e F_{\mu\nu} a_n.$$ (38)

we find (see also Ref. [28]),

$$[a_0] = 1,$$ (39)

$$[a_0|\mu] = 0,$$ (40)

$$[a_0|\mu\nu] = \frac{1}{2} i e F_{\mu\nu},$$ (41)

$$[a_0|\mu\nu\rho] = \frac{1}{3} i e (F_{\mu\nu,\rho} + F_{\mu\rho,\nu}),$$ (42)

and

$$[a_{0|\mu\nu\rho\tau}] = \frac{i e}{4} [F_{\mu\nu,\rho\tau} + F_{\mu\rho,\nu\tau} + F_{\mu\tau,\nu\rho}]$$

$$- \frac{e^2}{4} [F_{\mu\nu} F_{\rho\tau} + F_{\mu\rho} F_{\nu\tau} + F_{\mu\tau} F_{\nu\rho}]$$

$$+ \frac{i e}{4} [F_{\alpha\nu} S^\alpha_{\nu\rho\tau} + F_{\alpha\nu} S^\alpha_{\mu\tau\rho} + F_{\alpha\rho} S^\alpha_{\mu\tau\nu} + F_{\alpha\tau} S^\alpha_{\mu\rho\nu}].$$ (43)

Doing the same for $a_1$ setting $n = 0$ in Eq.(23), yields,

$$[a_1] = \left(\frac{1}{6} - \xi\right) R,$$ (44)

$$[a_{1|\mu}] = \frac{1}{2} \left(\frac{1}{6} - \xi\right) R_{;\mu} + \frac{i e}{6} F_{\mu\alpha} ;^\alpha,$$ (45)

and

$$[a_{1|\mu\nu}] = \left(\frac{1}{20} - \frac{1}{3} \xi\right) R_{\mu\nu} + \frac{1}{60} R_{\mu\nu,\alpha} + \frac{1}{90} R^{\alpha\beta} R_{\alpha\nu\beta\mu}$$

$$- \frac{1}{45} R_{\mu\alpha} R^\alpha_{\nu} + \frac{1}{90} R_{\alpha\beta\gamma\mu} R^{\alpha\beta\gamma}_{\nu} - \frac{e^2}{6} F_{\mu\alpha} F^\alpha_{\nu}$$

$$- \frac{i e}{12} (F_{\mu\alpha} ;^\alpha_{\nu} + F_{\nu\alpha} ;^\alpha_{\mu}) + \frac{i e}{2} \left(\frac{1}{6} - \xi\right) F_{\mu\nu}.$$ (46)

Finally, for $a_2$,
\[ [a_2] = -\frac{1}{180} R^{\alpha\beta} R_{\alpha\beta} + \frac{1}{180} R^{\alpha\beta\gamma\delta} R_{\alpha\beta\gamma\delta} + \frac{1}{6} (\frac{1}{5} - \xi) R_{\alpha\beta} + \frac{1}{2} (\frac{1}{6} - \xi)^2 R^2 - \frac{1}{12} F^{\alpha\beta} F_{\alpha\beta}. \] (47)

With these coincidence limits, we will be able to construct the quantities necessary to isolate the divergences in Eqs.\((8, 9)\).

V. COVARIANT EXPANSIONS

Evaluating the divergences of \(G^{(1)}\), the current, and the stress tensor requires expanding all of the terms in \(G^{(1)}\) and its derivatives in power series about the point \(x\) using the infinitesimal separation vector \(\sigma^\mu \equiv \sigma^\mu\). The generic form of the power series expansion of any bitensor \(a^{\mu_\nu...}(x, x')\) is

\[ a^{\mu_\nu...}(x, x') = a^{0\mu_\nu...} + a^{1\mu_\nu...} \sigma^\alpha + \frac{1}{2!} a^{2\mu_\nu...} \sigma^\alpha \sigma^\beta + \cdots. \] (48)

To evaluate the coefficients \(a^{0\mu_\nu...}, a^{1\mu_\nu...}, \ldots\), we take the coincidence limit \(x' \rightarrow x\) of Eq.\((48)\) and its derivatives. For example, for the second rank bitensor \(a^{\mu\nu}(x, x')\), this yields,

\[ a^{\mu\nu} = [a^{\mu\nu}], \] (49)
\[ a^{1\mu\nu}_\alpha = [a^{\mu\nu};\alpha] - a^{0\mu\nu;\alpha}, \] (50)
\[ a^{2\mu\nu}_{\alpha\beta} = [a^{\mu\nu};\alpha\beta] - 2a^{1\mu\nu}_{\alpha;\beta} - a^{0\mu\nu;\alpha\beta}, \] (51)
\[ a^{3\mu\nu}_{\alpha\beta\gamma} = [a^{\mu\nu};\alpha\beta\gamma] - 3a^{2\mu\nu}_{\alpha;\beta\gamma} - 3a^{1\mu\nu}_{\alpha;\beta\gamma} - a^{0\mu\nu;\alpha\beta\gamma}, \] (52)

and so forth. The numeric factors on the right-hand side of Eqs.\((50, 51, 52)\) arise due to symmetrization on the dummy indices \(\alpha, \beta, \ldots\), in Eq.\((48)\). Terms such as \(a^{1\mu\nu}_{\rho} \sigma^\rho \sigma^\alpha \sigma^\beta \sigma^\gamma\) do not contribute due to this same symmetrization.

The expansions will have bitensors constructed by taking primed derivatives of the biscalars in \(G^{(1)}\). These bitensors must be parallel transported back to the point \(x\) (see the form of Eq.\((3)\) and Eq.\((2)\)) in order to perform the expansion in Eq.\((48)\) (See Ref. \([2]\) for a more complete discussion of this point.) For example,
\[ g_{\mu'} \rho' a^{\mu'\rho'} = a0^{\mu'} + a1^{\mu'}_{\alpha} \sigma^{\alpha} + \cdots . \] (53)

To evaluate the coincidence limits in Eqs. (49-52) for these primed derivatives, we will use Christensen’s generalization [2] of a theorem proved by Synge,

\[ [T_{\alpha_1 \cdots \alpha_n \beta'_1 \cdots \beta'_m ; \mu}] = - [T_{\alpha_1 \cdots \alpha_n \beta_1' \cdots \beta_m ; \mu}] + [T_{\alpha_1 \cdots \alpha_n \beta'_1 \cdots \beta'_m ; : \mu}] , \] (54)

where \( T_{\alpha_1 \cdots \alpha_n \beta_1 \cdots \beta_m} \) is any bitensor with equal weight at both \( x \) and \( x' \) and whose coincidence limit and derivative coincidence limits exist. We will also use the relation

\[ [g_{\mu' ; \alpha \beta ; \cdots}] \sigma^\alpha \sigma^\beta \cdots = 0. \] (55)

Applying these to \( g_{\mu'}^{\nu} g_{\nu'}^{\rho} a^{\nu' \rho'} \), for example, yields,

\[
\begin{align*}
2^{\mu \nu}_{\alpha \beta} &= [(g_{\mu'}^{\nu} g_{\nu'}^{\rho} a^{\nu' \rho'})_{[\alpha \beta]} - a0^{\mu \nu}_{\alpha \beta} - a1^{\mu \nu}_{\alpha \beta} - a1^{\mu \nu}_{\beta \alpha}, \\
&= [a_{\alpha \beta}^{\mu \nu}] - [a_{\alpha \beta}^{\nu}]^{\mu} - [a_{\alpha \beta}^{\mu}]^{\nu} + [a_{\alpha \beta}]^{\mu \nu} - a0^{\mu \nu}_{\alpha \beta} \\
&\quad - a1^{\mu \nu}_{\alpha \beta} - a1^{\mu \nu}_{\beta \alpha} + [g^{\rho \sigma ; \alpha \beta}] \text{ terms,}
\end{align*}
\] (56)

where we have used the fact that primed derivatives commute with unprimed derivatives, and terms containing objects such as \( [g^{\mu \rho ; \alpha}] \), etc., have been grouped together since they contribute nothing to the expansions due to (55).

It can be seen from the expressions for \( G^{(1)} \), the current, and the stress tensor the order in \( \sigma^{\mu} \) to which each biscalar expansion must be carried. Applying this expansion method to the biscalars and their gauge covariant derivatives with respect to both \( x \) and \( x' \), we arrive at the following expansions,

\[ g_{\mu \nu}^{\mu} = -\sigma^{\mu} \], \( \sigma^{\mu \nu} = g^{\mu \nu} - \frac{1}{3} R^{\mu \nu}_{\alpha \beta} \sigma^{\alpha \beta} + \frac{1}{12} R^{\mu \nu}_{\alpha \beta \gamma \delta} \sigma^{\alpha \beta} \sigma^{\gamma \delta} + \cdots \], \( g_{\rho' \rho}^{\nu} \sigma^{\mu \nu} = -g^{\mu \nu} - \frac{1}{6} R^{\mu \nu}_{\alpha \beta} \sigma^{\alpha \beta} + \frac{1}{12} R^{\mu \nu}_{\alpha \beta \gamma \delta} \sigma^{\alpha \beta} \sigma^{\gamma \delta} + \cdots \)

(57) \( \sigma^{\nu} \), \( \sigma^{\mu \nu} \), \( g_{\nu}^{\rho'} \sigma^{\rho' \mu} \)
\[ g^{\mu\nu} g^{\rho\sigma} \sigma^{\mu\nu\rho\sigma} = g^{\mu\nu} - \frac{1}{3} R^\mu_{\alpha\nu} \sigma^\alpha \sigma^\beta + \frac{1}{4} R^\mu_{\alpha\nu} \beta\gamma \sigma^\alpha \sigma^\beta \sigma^\gamma \]

\[ \Delta^\pm = 1 + \frac{1}{12} R_{\alpha\beta} \sigma^\alpha \sigma^\beta - \frac{1}{24} R_{\alpha\beta} \gamma \sigma^\alpha \sigma^\beta \sigma^\gamma \]

\[ \Delta^{\mp\mu} = \frac{1}{6} R^\mu_{\alpha\sigma} \sigma^\alpha - \frac{1}{24} (2R^\mu_{\alpha\beta} - R_{\alpha\beta} \gamma) \sigma^\alpha \sigma^\beta + \left( \frac{1}{40} R^\mu_{\alpha\beta\gamma} \right) \sigma^\alpha \sigma^\beta \sigma^\gamma \]

\[ g^{\mu\nu} \Delta^{\mp\mu} = -\frac{1}{6} R^\mu_{\alpha\nu} + \frac{1}{12} (R^\mu_{\alpha\nu} - R^\nu_{\alpha\mu} - R^{\mu\nu};\alpha) \sigma^\alpha \]

\[ g^\nu_{\mu} \Delta^{\pm\mu} = \frac{1}{6} R^\mu_{\nu\rho} - \frac{1}{12} (R^\mu_{\nu\rho} - R^{\mu\nu};\rho) \sigma^\alpha \]

\[ g^{\mu\nu} \Delta^{\mp\nu\rho} = \frac{1}{6} R^\mu_{\nu\rho} - \frac{1}{12} (R^\mu_{\nu\rho} + R^{\mu\nu};\rho) \sigma^\alpha \]

\[ a_0 = 1, \]

\[ a_0^\mu = \frac{i e}{2} F^\mu_{\alpha\sigma} - \frac{i e}{6} F^\mu_{\alpha\beta} \sigma^\alpha \sigma^\beta + \frac{i e}{24} (F^\mu_{\alpha\beta\gamma} + F_{\rho\alpha} R^\mu_{\beta\gamma}) \sigma^\alpha \sigma^\beta \sigma^\gamma + \cdots, \]
\[ g^{\mu\nu}a_0^{\{l\}} = \frac{i e}{2} F^{\mu}_a \sigma^a - \frac{i e}{3} F^{\mu}_{a;\beta} \sigma^a \sigma^\beta + \frac{i e}{24} \left( 3 F^{\mu}_{a;\beta\gamma} + F_{\rho a} R^{\mu}_{\beta\gamma} \right) \sigma^{a\beta\gamma} + \cdots, \tag{69} \]

\[ a_0^{\{\mu\}} = \frac{i e}{2} F^{\mu} + \frac{i e}{6} \left( F^{\mu}_{a;\nu} + F^{\nu}_{a;\mu} \right) \sigma^a + \left( -\frac{i e}{24} F^{\mu}_{a;\nu} \sigma^{a\beta} \right) - \frac{e^2}{4} F^{\mu}_{a} F^a_{\beta} \]
\[ + \frac{i e}{12} \left( F^p_{\rho} R^p_{\alpha\beta} + F^p_{\nu} R^p_{\alpha\beta} \right) - \frac{i e}{24} F_{\rho a} \left( R^{\mu\nu}_{\beta\gamma} + R^{\nu\mu}_{\beta\gamma} \right) \sigma^{a\beta\gamma} \gamma + \cdots, \tag{70} \]

\[ g^{\nu\rho} a_0^{\{\mu\}} = -\frac{i e}{2} F^{\mu
u} + \frac{i e}{6} \left( F^{\mu
u}_{a;\nu} + F^{\mu
u}_{a;\nu} \right) \sigma^a + \left( -\frac{i e}{24} F^{\mu\nu}_{a;\nu} \sigma^{a\beta} \right) - \frac{e^2}{4} F^{\mu\nu}_{a} F^{\nu\rho}_{a} \]
\[ + \frac{i e}{12} \left( F^p_{\rho} R^p_{\alpha\beta} + F^p_{\nu} R^p_{\alpha\beta} \right) + \frac{i e}{24} F_{\rho a} \left( R^{\mu\nu}_{\beta\gamma} + R^{\nu\mu}_{\beta\gamma} \right) \sigma^{a\beta\gamma} \gamma + \cdots, \tag{71} \]

\[ g^{\mu\tau} g^{\nu\rho} a_0^{\{\tau\}} = \frac{i e}{2} F^{\mu
u} + \frac{i e}{6} \left( F^{\mu
u}_{a;\nu} + F^{\mu
u}_{a;\nu} \right) \sigma^a + \left( -\frac{i e}{24} F^{\mu\nu}_{a;\nu} \sigma^{a\beta} \right) - \frac{e^2}{4} F^{\mu\nu}_{a} F^{\nu\rho}_{a} \]
\[ + \frac{i e}{12} \left( F^p_{\rho} R^p_{\alpha\beta} + F^p_{\nu} R^p_{\alpha\beta} \right) + \frac{i e}{8} F_{\rho a} \left( R^{\mu\nu}_{\beta\gamma} + R^{\nu\mu}_{\beta\gamma} \right) \sigma^{a\beta\gamma} \gamma + \cdots, \tag{72} \]

\[ a_1 = (\frac{1}{6} - \xi) R + \left( -\frac{1}{2} \frac{1}{6} - \xi \right) R_{\alpha a} \sigma^a + \left( -\frac{1}{10} \frac{1}{90} R_{\rho a} R^\rho + \frac{1}{180} R^{\rho\sigma} R_{\rho a} + \frac{1}{120} R_{\alpha\beta\rho} \right) \]
\[ + \frac{1}{2} \frac{1}{6} - \xi \right) R^\mu - \frac{i e}{6} F^{\mu\rho} \]
\[ + \frac{1}{45} \left( R^\mu R^\rho - \frac{1}{90} R^{\rho\sigma} R^\mu R_{\rho\sigma} + \frac{1}{90} R^{\rho\sigma} R_{\rho\sigma} + \frac{1}{90} R^\mu_\rho R^\rho_\sigma \right) + \frac{1}{3} \left( \frac{1}{20} - \frac{1}{6} \frac{1}{6} - \xi \right) R^\mu \]
\[ + \frac{i e}{12} \left( F^\mu_{\rho\rho} + F^\nu_{\rho\rho} \right) + \frac{i e}{2} \left( \frac{1}{6} - \xi \right) R^\mu \]
\[ + \frac{i e}{12} \left( F^\mu_{\rho\rho} - F^\nu_{\rho\rho} \right) + \frac{i e}{2} \left( \frac{1}{6} - \xi \right) R^\mu \]
\[ + \frac{i e}{12} \left( F^\mu_{\rho\rho} + F^\nu_{\rho\rho} \right) + \frac{i e}{2} \left( \frac{1}{6} - \xi \right) R^\mu \]
All of the affected terms are gauge invariant and thus do not need the connection $A^\mu$ in their derivatives. The expansions with either two unprimed or two primed derivatives have been symmetrized according to,

$$a^{|\mu\nu} = \frac{1}{2}(a^{|\mu\nu} + a^{|\nu\mu} + \chi^{\mu\nu} a),$$

where $\chi^{\mu\nu} = i e F^{\mu\nu}$ for the $a_n$ and $\chi^{\mu\nu} = 0$ for the other biscalars. Note that quantities such as $g^{\nu'}(\sigma^{-1})^{\mu\nu'}$ may only have their numerators expanded using Eqs.(57-79) since they diverge in the coincidence limit necessary for applying the expansion iterations in Eqs.(59-62):

$$g^{\nu'}(\sigma^{-1})^{\mu\nu'} = g^{\nu'}(2\sigma^{-3}\sigma^{\mu\nu}\sigma^{\nu'}) - \sigma^{-2}\sigma^{\mu\nu'}$$

$$= -2\sigma^{-3}\sigma^{\mu\nu'} - \sigma^{-2}(-g^{\mu\nu} - \cdots).$$

VI. RESULTS

Substituting the expansions in Eqs.(57-79) for the biscalars into Eq.(24) and collecting terms of like powers of $\sigma^\mu$ yields,

$$4\pi^2 G^{(1)}(x,x') = \frac{2}{(\sigma^\rho\sigma_\rho)} + \left[m^2 - \frac{1}{6} - \xi \right] R[\gamma + \frac{1}{2}] \left[m^2 (\sigma^\rho\sigma_\rho)\right] - \frac{m^2}{2}$$

$$+ \frac{1}{6} R_{\alpha\beta} \sigma^\alpha\sigma_\beta (\sigma^\rho\sigma_\rho) + \frac{1}{2} m^2 \left[-\frac{1}{180} R^{\rho\tau} R_{\rho\tau} + \frac{1}{180} R^{\rho\tau\kappa\lambda} R_{\rho\tau\kappa\lambda} + \frac{1}{6} \left(\frac{1}{5} - \xi \right) R_{\rho\rho} \right]$$

$$+ \frac{1}{2} \left(\frac{1}{6} - \xi \right) R^2 + \Omega \left(\frac{1}{m^4}\right).$$

This is the same as the result derived by Christensen for a real scalar field.

We differentiate Eq.(24) to find $G^{(1)|\mu}$ and $G^{(1)|\mu'}$, using the bivector of parallel transport $g^{\mu\nu}$ to construct the expectation value of the current defined in Eq.(8). Substituting the appropriate expansions from Eqs.(57-79) and collecting like powers of $\sigma^\mu$ isolates the terms in Eq.(8) which will diverge linearly and those which will remain finite as the points are brought together.
\[
\langle j^\mu(x, x') \rangle_{\text{linear}} = \frac{1}{4\pi^2} \frac{e^2 \sigma^\alpha F^\mu_{\alpha\rho}}{(\sigma^\rho \sigma^\rho)}, \tag{83}
\]

and,
\[
\langle j^\mu(x, x') \rangle_{\text{finite}} = \frac{1}{4\pi^2} \frac{e^2 \sigma^\alpha \sigma^\beta F^\mu_{\alpha\beta}}{(2\sigma^\rho \sigma^\rho)} + \psi \left( \frac{1}{m^2} \right). \tag{84}
\]

Finally, we form \(G^{(1)}[\mu\nu], G^{(1)}[\mu'\nu], G^{(1)}[\mu'\nu],\) and \(G^{(1)}[\mu'\nu]\) by differentiating Eq.\((23)\). These are then used to form the components of the stress energy tensor of Eq.\((7)\). Substituting the expansions from Eqs.\((77\text{-}79)\), and collecting like powers of \(\sigma^\mu\), yields the following expressions, with the points split,

\[
\langle T^{\mu\nu} \rangle_{\text{quartic}} = \frac{1}{2\pi^2} \frac{1}{(\sigma^\rho \sigma^\rho)^2} \left[ g^{\mu\nu} - 4 \frac{\sigma^\mu \sigma^\nu}{(\sigma^\rho \sigma^\rho)} \right], \tag{85}
\]

\[
\langle T^{\mu\nu} \rangle_{\text{quadratic}} = \frac{1}{4\pi^2} \frac{1}{(\sigma^\rho \sigma^\rho)} \times \frac{1}{3} \left[ (R^{\mu}_{\alpha\beta} + R^{\nu}_{\alpha\beta}) \frac{\sigma^\alpha}{(\sigma^\rho \sigma^\rho)} - \frac{2}{3} R^{\alpha\beta} \frac{(\sigma^\alpha \sigma^\beta \sigma^\mu \sigma^\nu)}{(\sigma^\rho \sigma^\rho)^2} \right] - \frac{1}{2} R^{\mu\nu} \frac{1}{2} R \left[ g^{\mu\nu} - 2 \frac{\sigma^\mu \sigma^\nu}{(\sigma^\rho \sigma^\rho)} \right] + 2 \left( R^{\mu\nu} R^{\alpha\beta} - R^{\alpha\beta} g^{\mu\nu} \right) \frac{(\sigma^\rho \sigma^\rho)}{g^{\mu\nu}} \right], \tag{86}
\]

\[
\langle T^{\mu\nu} \rangle_{\text{linear}} = \frac{1}{4\pi^2} \times \frac{1}{12} \left[ \frac{R^{\mu\nu}}{4} - R g^{\mu\nu} \right] \frac{\sigma^\alpha}{(\sigma^\rho \sigma^\rho)} - \frac{1}{6} \left( R^{\mu}_{\alpha\beta} + R^{\nu}_{\alpha\beta} - \frac{1}{4} R^{\alpha\beta} g^{\mu\nu} \right) \frac{\sigma^\alpha \sigma^\beta \sigma^\gamma}{(\sigma^\rho \sigma^\rho)^2} \right] \left( \frac{1}{6} - \xi \right) \left[ \frac{R^{\mu\nu}}{4} - \frac{1}{4} R^{\alpha\beta} g^{\mu\nu} \right] \frac{\sigma^\rho \sigma^\rho}{g^{\mu\nu}} \right] + \left( \frac{1}{6} - \xi \right)^2 \left[ \frac{3}{4} R^{\alpha\gamma} \frac{\sigma^\gamma}{(\sigma^\rho \sigma^\rho)} \frac{g^{\mu\nu}}{g^{\mu\nu}} \right], \tag{87}
\]

\[
\langle T^{\mu\nu} \rangle_{\text{logarithmic}} = \frac{1}{4\pi^2} \frac{1}{2} \left[ \gamma + 4 \frac{1}{1} \left[ m^2 (\sigma^\rho \sigma^\rho) \right] \right] \times \frac{1}{60} \left( R^{\rho\mu\nu\tau} R_{\rho\tau} - \frac{1}{4} R^{\rho\tau} R_{\rho\tau} g^{\mu\nu} - \frac{1}{2} \frac{1}{6} - \xi \right) \left[ m^2 (R^{\rho\mu\nu} - \frac{1}{2} R g^{\mu\nu}) \right] + \frac{1}{120} \left( R^{\rho\mu\nu} \right)^{\rho} - \frac{1}{180} R (R^{\rho\mu\nu} - \frac{1}{4} R g^{\mu\nu}) - \frac{1}{360} R^{\mu\nu} - \frac{1}{720} R^{\mu\nu} \frac{1}{R^{\rho\mu\nu}} - \frac{1}{4} R g^{\mu\nu} \right] + \frac{1}{4} R^{\rho\mu\nu} - \frac{1}{4} R g^{\mu\nu} \right] - \frac{1}{8} \frac{1}{3} \frac{1}{12} F^{\rho\mu} F_{\rho\mu} + \frac{1}{48} g^{\mu\nu} F^{\rho\mu} F_{\rho\mu} \right], \tag{88}
\]
and

\[
\langle T_{\mu\nu}\rangle_{\text{finite}} = \langle T_{\mu\nu}\rangle_{\text{finite,Christensen}} + \\
\frac{e^2}{4\pi^2} \left[ \frac{\sigma^\mu\sigma^\nu}{12(\sigma^\rho\sigma_\rho)^2} \sigma^\alpha\sigma^\beta F_\alpha^\gamma F_\beta^\gamma + \frac{1}{12(\sigma^\rho\sigma_\rho)} (\sigma^\alpha\sigma^\mu F_\alpha^\beta F_\beta^\nu + \sigma^\alpha\sigma^\nu F_\alpha^\beta F_\beta^\mu) \right. \\
+ \frac{1}{(\sigma^\rho\sigma_\rho)} (\xi - \frac{1}{4}) \sigma^\alpha\sigma^\beta F_\alpha^\mu F_\beta^\nu + \frac{1}{(\sigma^\rho\sigma_\rho)} (\xi - \frac{5}{24}) g^\mu\nu \sigma^\alpha\sigma^\beta F_\alpha^\gamma F_\beta^\gamma \\
- \frac{1}{96} \left( g^\mu\nu - 2 \frac{\sigma^\mu\sigma^\nu}{(\sigma^\rho\sigma_\rho)} \right) F_\alpha^\beta F_\alpha^\beta + \vartheta \left( \frac{1}{m^2} \right) \right].
\]

(89)

For brevity, only the terms due to the electromagnetic field are included in Eq. (89). The terms \( \langle T_{\mu\nu}\rangle_{\text{finite,Christensen}} \) may be found in Ref. [2].

VII. DISCUSSION

The complex scalar field is constructed from two real fields according to \( \phi(x) = \frac{1}{\sqrt{2}} (\phi_1(x) + i\phi_2(x)) \). From the form of the Hadamard function Eq. (7a), it is evident that \( G^{(1)} \) for the complex scalar field will be the sum of \( G^{(1)} \) for two real scalar fields. Thus, the agreement of Eq. (82) with Christensen’s original result is to be expected, with both containing the same quadratic and logarithmic divergences, with their corresponding direction-dependent (\( \sigma^\mu \)-dependent) and direction-independent terms.

The divergence of Eqs. (83) and (84) cannot be taken directly in order to verify the point-splitting procedure does not violate the conservation of current. This would involve taking the divergence with respect to \( x^\alpha \) of the separation vector \( \sigma^\mu(x, x') \), which depends on both \( x^\alpha \) and \( x'^\alpha \). The correct procedure for verifying conservation of current requires evaluating the divergence of the classical current within the point-splitting regime. The divergence of the classical current,

\[
j^\mu \mu = i e [(D^\mu \phi)\phi^* - \phi(D^\mu \phi)^*]_{\mu},
\]

(90)

may be written as,

\[
j^\mu \mu = \frac{ie}{2} \left[ \{D_\mu D^\mu \phi, \phi^*\} - \{D_\mu D^\mu \phi, \phi^*\}^* \right] = \frac{ie}{2} \left[ \{\phi^\mu_{\mu}, \phi^*\} - \{\phi^\mu_{\mu}, \phi^*\}^* \right].
\]

(91)
Making the transition from classical to quantum fields and applying the point-splitting procedure to the right-hand-side of Eq. (91) yields the proper expression to examine, namely, 

$$\langle j^\mu(x) \rangle = \lim_{x' \to x} \frac{i e}{4} \left[ (G^{(1)})^\mu_\mu + g^\mu_\nu g_\mu_\rho G^{(1)}_\nu_\rho \right] - \left( (G^{(1)})^\mu_\mu + g^\mu_\nu g_\mu_\rho G^{(1)}_\nu_\rho \right) \ast \right]. \tag{92}$$

This may be seen to vanish to all orders by using the expansions in Eqs. (57-79), ensuring that the current is a conserved quantity.

The linear divergence of Eq. (83) is to be expected by a straightforward analysis of the divergences present in scalar electrodynamics [29,30]. The interaction terms in the Lagrangian for the charged scalar field, $L_I \equiv j^\mu A_\mu$, may be written in the form,

$$L_I = \frac{ie}{2} \left[ \{\partial^\mu \phi, \phi^*\} - \{\partial^\mu \phi, \phi^*\} \ast \right] A_\mu - e^2 A^\mu A_\mu \{\phi, \phi^*\}. \tag{93}$$

The first term gives rise to a 3-point Feynman graph, while the second term has a 4-point graph. The degree of divergence $D$ present in scalar electrodynamics is given by,

$$D = 4 - P_e - Q_e, \tag{94}$$

where $P_e$ and $Q_e$ are the number of external scalar and photon lines, respectively. The values $D = 0, 1, 2, \ldots$, imply logarithmic, linear, quadratic,\ldots, divergences, while $D < 0$ implies the interaction is finite. DeWitt has pointed out for the generalized Yang-Mills field in the presence of the gravitational field that the simplest possible counterterm will always be the most divergent [26]. The simpler term represented by the 3-point graph indicates a degree of divergence $D = 1$. This linear divergence is in contrast to the well-known logarithmic divergence of the fermion field current [21]. The fermion field has an interaction Lagrangian given by,

$$L_I = \frac{e}{2} \overline{\psi} \gamma^\mu \gamma^\rho \psi A_\mu, \tag{95}$$

where the sum over the indices of the spinors and the gamma matrices is understood. The degree of divergence for spinor electrodynamics is given by,

$$D = 4 - \frac{3}{2} F_e - Q_e, \tag{96}$$
where $F_e$ and $Q_e$ are the number of external fermion and photon lines, respectively. The single 3-point graph indicated by (95) would thus have $D = 0$, an expected logarithmic divergence.

The current and stress energy tensor are independent of the sign of the charge carrier with both having electromagnetic terms proportional to $e^2$. One factor of $e$ in the current is explicit in Eq. (8). The other factor comes from the derivation of the expansions of $G(1)^{\mu}$, etc. This originates with the gauge commutation relation, Eq. (38), and is seen to carry one factor of $eF$ through to the first derivative expansions of the $a_n$ in Eqs. (57-79). The current is a third order quantity with units of $(\text{length})^{-3}$. By counting powers, the linearly diverging term would thus require one factor of the second order $eF$, while the finite term would require the third order derivative of $eF$. Eqs. (83) and (84) are thus in the only possible form and, aside from numerical factors, could have been deduced immediately from simple power counting.

The structure of the electromagnetic counterterms for the stress energy tensor may be deduced in like manner. They must be independent of the sign of the charge and thus proportional to $e^2$. There are no explicit factors of $e$ in Eq. (9), so they must arise in the derivations of the expansions of $G(1)^{\mu\nu}$, etc., by virtue of the gauge commutator. The simplest term expected is the fourth order $(eF)^2$. The fourth order stress energy tensor would only allow such a term to be present in the logarithmic counterterm. This is in agreement with DeWitt’s result for the generalized Yang-Mills field [26]. The two possible fourth order combinations of the gravitational and electromagnetic field tensors are $g^{\mu\nu} e^2 F^\rho F_\rho$ and $e^2 F^{\rho\mu} F_\rho^{\nu}$. The point-splitting procedure yields the numerical factors for these terms.

It is to be expected that some infinite counterterms involving only the electromagnetic field would remain even in the flat space limit. The present work yields some of those terms. It is not surprising that there are no cross terms here involving the electromagnetic field and curvature. The lowest order of such cross terms would be sixth order. The point-splitting procedure is an expansion in inverse powers of $m$ [26] and the current and stress energy tensor in the present case are truncated at order $m^0$. Sixth order cross terms such
as $(eF)^2 R$ are expected when the expansion is carried out to order $m^{-2}$ [31].

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